

# THE DEGREE AND REGULARITY OF VANISHING IDEALS OF ALGEBRAIC TORIC SETS OVER FINITE FIELDS

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**ABSTRACT.** Let  $X^*$  be a subset of an affine space  $\mathbb{A}^s$ , over a finite field  $K$ , which is parameterized by the edges of a clutter. Let  $X$  and  $Y$  be the images of  $X^*$  under the maps  $x \mapsto [x]$  and  $x \mapsto [(x, 1)]$  respectively, where  $[x]$  and  $[(x, 1)]$  are points in the projective spaces  $\mathbb{P}^{s-1}$  and  $\mathbb{P}^s$  respectively. For certain clutters and for connected graphs, we were able to relate the algebraic invariants and properties of the vanishing ideals  $I(X)$  and  $I(Y)$ . In a number of interesting cases, we compute its degree and regularity. For Hamiltonian bipartite graphs, we show the Eisenbud-Goto regularity conjecture. We give optimal bounds for the regularity when the graph is bipartite. It is shown that  $X^*$  is an affine torus if and only if  $I(Y)$  is a complete intersection. We present some applications to coding theory and show some bounds for the minimum distance of parameterized linear codes for connected bipartite graphs.

## 1. INTRODUCTION

Let  $K = \mathbb{F}_q$  be a finite field with  $q \neq 2$  elements and let  $y^{v_1}, \dots, y^{v_s}$  be a finite set of monomials. As usual if  $v_i = (v_{i1}, \dots, v_{in}) \in \mathbb{N}^n$ , then we set

$$y^{v_i} = y_1^{v_{i1}} \cdots y_n^{v_{in}}, \quad i = 1, \dots, s,$$

where  $y_1, \dots, y_n$  are the indeterminates of a ring of polynomials with coefficients in  $K$ . Consider the following sets parameterized by these monomials: (a) the *affine algebraic toric set*

$$X^* := \{(x_1^{v_{11}} \cdots x_n^{v_{1n}}, \dots, x_1^{v_{s1}} \cdots x_n^{v_{sn}}) \in \mathbb{A}^s \mid x_i \in K^* \text{ for all } i\},$$

where  $K^* = \mathbb{F}_q^* = \mathbb{F}_q \setminus \{0\}$  and  $\mathbb{A}^s = K^s$  is an affine space over the field  $K$ , (b) the *projective algebraic toric set*

$$X := \{[(x_1^{v_{11}} \cdots x_n^{v_{1n}}, \dots, x_1^{v_{s1}} \cdots x_n^{v_{sn}})] \mid x_i \in K^* \text{ for all } i\} \subset \mathbb{P}^{s-1},$$

where  $\mathbb{P}^{s-1}$  is a projective space over the field  $K$ , and (c) the *projective closure* of  $X^*$

$$Y := \{[(x_1^{v_{11}} \cdots x_n^{v_{1n}}, \dots, x_1^{v_{s1}} \cdots x_n^{v_{sn}}, 1)] \mid x_i \in K^* \text{ for all } i\} \subset \mathbb{P}^s.$$

Notice that  $Y$  is parameterized by  $y^{v_1}, \dots, y^{v_s}, y^{v_{s+1}}$ , where  $v_{s+1} = 0$ . These three sets are multiplicative groups under componentwise multiplication. We are interested in the algebraic invariants (regularity, degree, Hilbert series)—and in the complete intersection property—of the vanishing ideals of these sets.

Let  $S = K[t_1, \dots, t_s] = \bigoplus_{d=0}^{\infty} S_d$  and  $S[u] = \bigoplus_{d=0}^{\infty} S_d[u]$  be polynomial rings over the field  $K$  with the standard grading, where  $S[u]$  is obtained from  $S$  by adjoining a new variable  $u = t_{s+1}$ .

Recall that the *vanishing ideal* of  $X^*$ , denoted by  $I(X^*)$ , is the ideal of  $S$  generated by all polynomials that vanish on  $X^*$ . The *vanishing ideal* of  $X$  (resp.  $Y$ ), denoted by  $I(X)$  (resp.

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$I(Y)$ ), is the ideal of  $S$  (resp.  $S[u]$ ) generated by the homogeneous polynomials that vanish on  $X$  (resp.  $Y$ ).

In this paper we uncover some relationships between the algebraic invariants—and the complete intersection properties—of  $I(X)$  and  $I(Y)$ . We focus on vanishing ideals of algebraic toric sets that are parameterized by monomials  $y^{v_1}, \dots, y^{v_s}$  arising from the edges of a graph  $G$  or clutter  $\mathcal{C}$  (a clutter is a sort of hypergraph, see Definition 2.1).

This paper is motivated by the study of parameterized linear codes [25], and specifically by the fact that the degree and the Hilbert function of  $S[u]/I(Y)$  are related to the basic parameters of parameterized affine linear codes [22] (see Theorem 3.4).

The contents of this paper are as follows. In Section 2 we study the degree and regularity of vanishing ideals. It is well known that  $|X|$  and  $|Y|$  are the degrees of  $S/I(X)$  and  $S[u]/I(Y)$  respectively [20]. We show that  $|Y| \leq (q-1)|X|$  and give sufficient conditions for equality in terms of  $q$  and the combinatorics of  $\mathcal{C}$  (see Proposition 2.5). If  $G$  is a graph, we express  $|Y|$  as a function of  $q$ ,  $n$  and  $|X|$  (see Theorem 2.8). For connected graphs, we express  $|Y|$  as a function of  $q$  and  $n$  only (Corollary 2.9). In general the ideal  $I(X) + (t_1^{q-1} - u^{q-1})$  is contained in  $I(Y)$ . We give sufficient conditions for equality (see Theorem 2.10), for instance equality occurs if  $G$  is a bipartite graph or if  $G$  is any graph and  $q$  is even (see Corollary 2.11). It turns out that the invariants of  $S/I(X)$  and  $S[u]/I(Y)$  are closely related if equality occurs (see Proposition 2.12). For connected bipartite graphs, we give optimal upper and lower bounds for the regularity of  $S/I(X)$  (see Theorem 2.18). Then, we compute the regularity of any Hamiltonian bipartite graph (see Corollary 2.21). As a byproduct, we show the Eisenbud-Goto regularity conjecture when  $G$  is a Hamiltonian bipartite graph (see Corollary 2.24). Let  $X'$  be the set parameterized by  $y^{v_1}, \dots, y^{v_{s-1}}$ . If  $y_n$  occurs only in the monomial  $y^{v_s}$ , we relate the degree and the regularity of  $I(X)$  and  $I(X')$  (see Theorem 2.27). For connected bipartite graphs, this leads to an improved upper bound for the regularity of  $S/I(X)$ , in terms of the length of a largest cycle (see Corollary 2.31).

In Section 3, we give applications to coding theory, and explain the well known connections between the *algebraic invariants* of vanishing ideals (Hilbert function, degree, regularity) and the *parameters* of affine and projective parameterized linear codes (dimension, length, minimum distance). We present upper and lower bounds for the minimum distance of parameterized codes arising from connected bipartite graphs (see Theorem 3.6). The bounds are in terms of the minimum distance of parameterized codes over projective tori. These bounds can be computed using a recent result of [26] (see Theorem 3.7). Let  $\delta_Y(d)$  (resp.  $\delta_X(d)$ ) be the minimum distance of the parameterized projective code of degree  $d$  on the set  $Y$  (resp.  $X$ ), see Definition 3.2. For certain clutters we show that  $\delta_Y(d) \leq (q-1)\delta_X(d)$  for  $d \geq 1$  with equality if  $d = 1$  and  $G$  is a connected bipartite graph (see Proposition 3.10).

In Section 4, we characterize when  $I(Y)$  is a complete intersection in algebraic and geometric terms (see Theorem 4.5). A result of [26] shows that  $I(X)$  is a complete intersection if and only if  $X$  is a projective torus (see Definition 2.15). We complement this result by showing that  $I(Y)$  is a complete intersection if and only if  $X^*$  is an affine torus (see Theorem 4.5). For connected graphs, the complete intersection property of  $I(X)$  is independent of  $q$  (see Proposition 4.9), while the complete intersection property of  $I(Y)$  depends on  $q$ . We describe when  $I(Y)$  is a complete intersection in terms of  $q$  and the combinatorics of the graph (see Theorem 4.10).

For all unexplained terminology and additional information we refer to [23] (for the general theory of commutative rings), [1, 29] (for the theory of Gröbner bases and Hilbert functions),

[12, 16, 33] (for the theory of Reed-Muller codes and evaluation codes), [25] (for the theory of parameterized codes), and [4, 6] (for graph theory and clutter theory).

## 2. THE DEGREE AND THE REGULARITY OF VANISHING IDEALS

We continue to use the notation and definitions used in the introduction. In this section we study the degree and the regularity of  $S/I(X)$  and  $S[u]/I(Y)$ .

**Definition 2.1.** A *clutter*  $\mathcal{C}$  is a family  $E$  of subsets of a finite ground set  $\{y_1, \dots, y_n\}$  such that if  $f_1, f_2 \in E$ , then  $f_1 \not\subset f_2$ . The ground set is called the *vertex set* of  $\mathcal{C}$  and  $E$  is called the *edge set* of  $\mathcal{C}$ , they are denoted by  $V_{\mathcal{C}}$  and  $E_{\mathcal{C}}$  respectively.

Clutters are special hypergraphs. One important example of a clutter is a graph with the vertices and edges defined in the usual way for graphs [4].

**Definition 2.2.** Let  $\mathcal{C}$  be a clutter with vertex set  $V_{\mathcal{C}} = \{y_1, \dots, y_n\}$  and let  $f$  be an edge of  $\mathcal{C}$ . The *characteristic vector* of  $f$  is the vector  $v = \sum_{y_i \in f} e_i$ , where  $e_i$  is the  $i$ th unit vector in  $\mathbb{R}^n$ .

Throughout this paper  $\mathcal{C}$  will denote a clutter with  $n$  vertices and  $s$  edges. We will always assume that  $\{v_1, \dots, v_s\}$  is the set of all characteristic vectors of the edges of  $\mathcal{C}$ . We also assume that  $y_1, \dots, y_n$  are the vertices of  $\mathcal{C}$ . When  $\mathcal{C}$  is a graph, we denote  $\mathcal{C}$  by  $G$ .

**Definition 2.3.** Let  $\mathcal{C}$  be a clutter. We call  $X$  (resp.  $X^*$ ) the *projective algebraic toric set* (resp. *affine algebraic toric set*) parameterized by the edges of  $\mathcal{C}$ .

**Definition 2.4.** The *Hilbert function* of  $S[u]/I(Y)$  is given by

$$H_Y(d) := \dim_K (S[u]/I(Y))_d = \dim_K S[u]_d / I(Y)_d,$$

where  $I(Y)_d = S[u]_d \cap I(Y)$  is the degree  $d$  part of  $I(Y)$ .

The ideal  $I(Y)$  is Cohen-Macaulay of height  $s$  [11]. Thus,  $\dim S[u]/I(Y) = 1$ . The unique polynomial  $h_Y(t) \in \mathbb{Z}[t]$  such that  $h_Y(d) = H_Y(d)$  for  $d \gg 0$  is called the *Hilbert polynomial* of  $S[u]/I(Y)$ . In our situation  $h_Y(t)$  is a constant. Furthermore  $H_Y(d) = |Y|$  for  $d \geq |Y| - 1$ , see [20, Lecture 13]. This means that  $|Y|$  is the *degree* of  $S[u]/I(Y)$ . Likewise, the integer  $|X|$  is the degree of  $S/I(X)$ . The *index of regularity* of  $S[u]/I(Y)$ , denoted by  $\text{reg}(S[u]/I(Y))$ , is the least integer  $p \geq 0$  such that  $h_Y(d) = H_Y(d)$  for  $d \geq p$ . Under our hypothesis, the index of regularity of  $S[u]/I(Y)$  is the Castelnuovo-Mumford regularity of  $S[u]/I(Y)$  [9]. We will refer to  $\text{reg}(S[u]/I(Y))$  simply as the *regularity* of  $S[u]/I(Y)$ .

We shall be interested in computing the degree and the regularity of  $S[u]/I(Y)$  and  $S/I(X)$  in terms of the invariants of the clutter  $\mathcal{C}$  and the number of elements of the field  $K$ .

Let  $k \geq 2$  be an integer. A clutter is called *k-uniform* if all its edges have cardinality  $k$ .

**Proposition 2.5.** *Let  $\mathcal{C}$  be a clutter.*

- (i)  $|Y| \leq (q-1)|X|$ .
- (ii) *If there is  $C \subset V_{\mathcal{C}}$  so that  $|C \cap e| = 1$  for any  $e \in E_{\mathcal{C}}$ , then  $|Y| = (q-1)|X|$ .*
- (iii) *If  $\mathcal{C}$  is a  $k$ -uniform clutter and  $\gcd(q-1, k) = 1$ , then  $|Y| = (q-1)|X|$ .*

*Proof.* (i) Let  $\mathbb{T}' = \{[(z_1, \dots, z_{s+1})] \mid z_i \in K^*\}$  and  $\mathbb{T} = \{[(z_1, \dots, z_s)] \mid z_i \in K^*\}$  be two projective torus in  $\mathbb{P}^s$  and  $\mathbb{P}^{s-1}$  respectively. The projection map

$$\mathbb{T}' \longrightarrow \mathbb{T}, \quad [(z_1, \dots, z_{s+1})] \longmapsto [(z_1, \dots, z_s)],$$

induces an epimorphism of multiplicative groups  $\theta: Y \rightarrow X$ . By the fundamental homomorphism theorem for groups one has an isomorphism  $Y/\ker(\theta) \simeq X$ . Since we have the inclusion

$$\ker(\theta) \subset \{[(u, \dots, u, 1)] \mid u \in K^*\},$$

we get  $|Y| = |\ker(\theta)||X| \leq (q-1)|X|$ . This completes the proof of (i).

(ii) We may assume that  $C = \{y_1, \dots, y_\ell\}$ . Let  $[(x^{v_1}, \dots, x^{v_s})]$  be a point in  $X$  and let  $\gamma$  be an arbitrary element of  $K^*$ . From the equality

$$[(x^{v_1}, \dots, x^{v_s}, \gamma)] = \left[ \left( \left( \frac{x_1}{\gamma} \right)^{v_{11}} \dots \left( \frac{x_\ell}{\gamma} \right)^{v_{1\ell}} x_{\ell+1}^{v_{1,\ell+1}} \dots x_n^{v_{1n}}, \dots, \left( \frac{x_1}{\gamma} \right)^{v_{s1}} \dots \left( \frac{x_\ell}{\gamma} \right)^{v_{s\ell}} x_{\ell+1}^{v_{s,\ell+1}} \dots x_n^{v_{sn}}, 1 \right) \right]$$

we get that  $[(x^{v_1}, \dots, x^{v_s}, \gamma)] \in Y$ . Let  $\beta$  be a generator of the cyclic group  $(\mathbb{F}_q^*, \cdot)$ . We can choose  $P_1, \dots, P_m$  in  $X^*$  such that  $X = \{[P_1], \dots, [P_m]\}$ . Hence, the set

$$\{[(P_1, \beta)], \dots, [(P_1, \beta^{q-1})], \dots, [(P_m, \beta)], \dots, [(P_m, \beta^{q-1})]\}$$

is contained in  $Y$  and has exactly  $(q-1)|X|$  elements. Therefore  $|Y| \geq (q-1)|X|$ . The reverse inequality follows from (i).

(iii) The map  $\mathbb{F}_q^* \rightarrow \mathbb{F}_q^*$ ,  $a \mapsto a^k$ , is an isomorphism of multiplicative groups if and only if  $\gcd(q-1, k) = 1$ . For each  $a \in \mathbb{F}_q^*$ , making  $x_i = a$  for all  $i$  in  $[(x^{v_1}, \dots, x^{v_s}, 1)]$ , we get that the point  $[(a^k, \dots, a^k, 1)]$  is in the kernel of  $\theta$ . Thus  $|Y| = |\ker(\theta)||X| \geq (q-1)|X|$ . The reverse inequality follows from (i).  $\square$

**Definition 2.6.** A graph  $G$  is called *bipartite* if its vertex set can be partitioned into two disjoint subsets  $V_1$  and  $V_2$  such that every edge of  $G$  has one end in  $V_1$  and one end in  $V_2$ . The pair  $(V_1, V_2)$  is called a *bipartition* of  $G$ .

**Remark 2.7.** If  $G$  is a connected bipartite graph, there is only one bipartition of  $G$ .

We come to the first main result of this section.

**Theorem 2.8.** *If  $G$  is a graph, then*

$$|Y| = \begin{cases} (i) (q-1)|X|, & \text{if } G \text{ is bipartite.} \\ (ii) (q-1)|X|, & \text{if } \gcd(q-1, 2) = 1. \\ (iii) \frac{(q-1)}{2}|X|, & \text{if } G \text{ is not bipartite and } \gcd(q-1, 2) \neq 1. \end{cases}$$

*Proof.* Let  $(V_1, V_2)$  be a bipartition of  $G$ . Notice that the set  $C = V_1$  satisfies that  $|C \cap e| = 1$  for any  $e \in E_G$ . Thus, (i) and (ii) follow from Proposition 2.5.

(iii) We set  $L = \{[(a^2, \dots, a^2, 1)] \mid a \in \mathbb{F}_q^*\} \subset \mathbb{P}^s$ . The projection map

$$\{[(z_1, \dots, z_{s+1})] \mid z_i \in K^*\} \rightarrow \{[(z_1, \dots, z_s)] \mid z_i \in K^*\}, \quad [(z_1, \dots, z_{s+1})] \mapsto [(z_1, \dots, z_s)],$$

induces an epimorphism of multiplicative groups  $\theta: Y \rightarrow X$ . We claim that  $\ker(\theta) = L$ . The inclusion “ $\supset$ ” clearly holds and is true for any graph. To show the other inclusion we proceed by contradiction. Pick  $[P] \in \ker(\theta) \setminus L$ . This means that  $[P] = [(b, \dots, b, 1)]$  for some  $b \in \mathbb{F}_q^*$  and  $b \neq a^2$  for any  $a \in \mathbb{F}_q^*$ . Since  $G$  is not a bipartite graph,  $G$  contains an odd cycle  $\mathcal{C}_k = \{y_1, \dots, y_k\}$  of length  $k$ . We may assume that  $y^{v_1}, \dots, y^{v_k}$  are the monomials that correspond to the edges of the cycle  $\mathcal{C}_k$ . Thus, any element of  $Y$  is of the form

$$[(x_1 x_2, x_2 x_3, \dots, x_{k-1} x_k, x_1 x_k, x^{v_{k+1}}, \dots, x^{v_s}, 1)]$$

with  $x_i \in \mathbb{F}_q^*$  for all  $i$ . Since the kernel of  $\theta$  is given by

$$\ker(\theta) = \{[(u, \dots, u, 1)] \mid u \in \mathbb{F}_q^*\} \cap Y$$

and since  $[P]$  is in the kernel of  $\theta$ , we can write

$$b = x_1x_2 = x_2x_3 = \cdots = x_{k-1}x_k = x_1x_k$$

for some  $x_1, \dots, x_k$  in  $\mathbb{F}_q^*$ . Hence

$$x_1 = x_3 = \cdots = x_k \quad \text{and} \quad x_2 = x_4 = \cdots = x_{k-1} = x_1.$$

Thus,  $b = x_i^2$  for  $i = 1, \dots, k$ , a contradiction. This completes the proof of the claim. Next, we prove the equality  $|L| = (q-1)/2$ . Let  $\beta$  be a generator of the cyclic group  $(\mathbb{F}_q^*, \cdot)$ . In this case the image of the map  $\mathbb{F}_q^* \rightarrow \mathbb{F}_q^*$ ,  $a \mapsto a^2$ , is a subgroup of  $\mathbb{F}_q^*$  of order  $(q-1)/2$  because  $\beta^2$  is a generator of the image and this element has order  $(q-1)/2$ . Therefore,  $|L| = (q-1)/2$ . Hence, from the isomorphism  $Y/\ker(\theta) \simeq X$  and using that  $L = \ker(\theta)$ , we get  $|Y| = \frac{(q-1)}{2}|X|$ .  $\square$

**Corollary 2.9.** *Let  $G$  be a connected graph. Then*

$$|Y| = \begin{cases} \text{(i)} & (q-1)^{n-1}, & \text{if } G \text{ is bipartite.} \\ \text{(ii)} & (q-1)^n, & \text{if } G \text{ is not bipartite and } q \text{ is even.} \\ \text{(iii)} & \frac{(q-1)^n}{2}, & \text{if } G \text{ is not bipartite and } q \text{ is odd.} \end{cases}$$

*Proof.* From [25], one has that  $|X| = (q-1)^{n-2}$  if  $G$  is bipartite and  $|X| = (q-1)^{n-1}$  otherwise. Hence, the result follows from Theorem 2.8.  $\square$

Let  $I$  be an ideal of  $S$  and let  $S' = S[u]$ . By abuse of notation, we will write  $I$  in place of  $IS'$  when it is clear from context that we are using the generators of  $I$  but extending to an ideal of the larger ring  $S'$ .

**Theorem 2.10.** *Let  $\mathcal{C}$  be a clutter.*

- (a) *If there is  $C \subset V_{\mathcal{C}}$  so that  $|C \cap e| = 1$  for any  $e \in E_{\mathcal{C}}$ , then  $I(Y) = I(X) + (t_1^{q-1} - t_{s+1}^{q-1})$ .*
- (b) *If  $\mathcal{C}$  is a  $k$ -uniform clutter and  $\gcd(q-1, k) = 1$ , then  $I(Y) = I(X) + (t_1^{q-1} - t_{s+1}^{q-1})$ .*

*Proof.* We set  $I' = I(X) + (t_1^{q-1} - t_{s+1}^{q-1})$ . Notice that  $I' = I(X) + (t_i^{q-1} - t_{s+1}^{q-1})$  for any  $1 \leq i \leq s$ . Clearly  $I' \subset I(Y)$ . To show the reverse inclusion we proceed by contradiction. Assume there is a homogeneous polynomial  $f \in I(Y) \setminus I'$ . The ideal  $I(Y)$  is a lattice ideal [25, Theorem 2.1], i.e.,  $I(Y)$  is a binomial ideal and  $t_i$  is not a zero divisor of  $S[u]/I(Y)$  for all  $i$ . Thus, we may assume that  $f$  is a binomial which is a minimal generator of  $I(Y)$ . Hence, we can write

$$(2.1) \quad f = t_1^{a_1} \cdots t_s^{a_s} t_{s+1}^{a_{s+1}} - t_1^{b_1} \cdots t_s^{b_s} t_{s+1}^{b_{s+1}},$$

such that for each  $1 \leq j \leq s+1$  either  $a_j = 0$  or  $b_j = 0$ . We may also assume that  $a_{s+1} = 0$ ,  $b_{s+1} > 0$ ,  $a_i > 0$ ,  $b_i = 0$  for some  $i$ . For simplicity we assume that  $i = 1$ . We can choose  $f$  of least possible degree, i.e., any binomial in  $I(Y)$  of degree less than  $\deg(f)$  belongs to  $I'$ . Let  $\beta$  be a generator of the cyclic group  $(\mathbb{F}_q^*, \cdot)$ .

(a) We may assume that  $C = \{y_1, \dots, y_\ell\}$ . Making  $x_i = \beta^{-1}$  for  $1 \leq i \leq \ell$  and  $x_i = 1$  for  $i > \ell$  in  $[(x^{v_1}, \dots, x^{v_s}, 1)]$ , we get

$$\begin{aligned} [(x^{v_1}, \dots, x^{v_s}, 1)] &= [((\beta^{-1})^{v_{11}} \cdots (\beta^{-1})^{v_{1\ell}}, \dots, (\beta^{-1})^{v_{s1}} \cdots (\beta^{-1})^{v_{s\ell}}, 1)] \\ &= [(\beta^{-1}, \dots, \beta^{-1}, 1)] = [(1, \dots, 1, \beta)]. \end{aligned}$$

Thus,  $[(1, \dots, 1, \beta)] \in Y$ . Then, from Eq. (2.1) and using that  $f$  vanishes on  $Y$ , we get that  $\beta^{b_{s+1}} = 1$ . Thus,  $b_{s+1} = r(q-1)$  for some integer  $r$ . From the equality

$$\begin{aligned} f - t_{s+1}^{(r-1)(q-1)} t_1^{b_1} \cdots t_s^{b_s} (t_1^{q-1} - t_{s+1}^{q-1}) \\ = (t_1^{a_1} \cdots t_s^{a_s} t_{s+1}^{a_{s+1}} - t_{s+1}^{(r-1)(q-1)} t_1^{b_1+(q-1)} t_2^{b_2} \cdots t_s^{b_s}) = t_1 h, \end{aligned}$$

we obtain that the binomial  $h$  is homogeneous, belongs to  $I(Y)$ , and has degree less than  $\deg(f)$ . Thus,  $h \in I'$ . Consequently  $f \in I'$ , a contradiction.

(b) Making  $x_i = \beta$  for all  $i$  in  $[(x^{v_1}, \dots, x^{v_s}, 1)]$  we obtain that  $[(\beta^k, \dots, \beta^k, 1)]$  is in  $Y$ . Then, using that  $f$  vanishes on  $Y$  together with Eq. (2.1), we get that  $\beta^{kb_{s+1}} = 1$ . As  $k$  and  $q-1$  are relatively prime, we obtain that  $b_{s+1} = r(q-1)$  for some integer  $r$ . Hence, we may proceed as in (a) to derive a contradiction.  $\square$

**Corollary 2.11.** *Let  $G$  be a graph. If  $G$  is bipartite or if  $\gcd(q-1, 2) = 1$ , then*

$$I(Y) = I(X) + (t_1^{q-1} - t_{s+1}^{q-1}).$$

*Proof.* If  $G$  is bipartite, pick a bipartition  $(V_1, V_2)$  of  $G$ . Then, the set  $C = V_1$  satisfies that  $|C \cap e| = 1$  for any  $e \in E_G$ . Thus, the equality follows from Theorem 2.10(a). If  $\gcd(q-1, 2) = 1$ , the equality follows from Theorem 2.10(b) because any graph is 2-uniform.  $\square$

The degree and the regularity of  $S/I(X)$  can be computed using Hilbert series as we now explain. The Hilbert series  $F_X(t)$  of  $S/I(X)$  can be written as

$$F_X(t) := \sum_{i=0}^{\infty} H_X(i)t^i = \sum_{i=0}^{\infty} \dim_K(S/I(X))_i t^i = \frac{h_0 + h_1 t + \dots + h_r t^r}{1-t},$$

where  $h_0, \dots, h_r$  are positive integers (see [29]). This follows from the fact that  $S/I(X)$  is a Cohen-Macaulay standard algebra of dimension 1 [11]. The number  $r$  is the regularity of  $S/I(X)$  and  $h_0 + \dots + h_r$  is the degree of  $S/I(X)$  (see [29] or [34, Corollary 4.1.12]).

**Proposition 2.12.** *Let  $F_X(t)$  and  $F_Y(t)$  be the Hilbert series of  $S/I(X)$  and  $S[u]/I(Y)$  respectively. If  $I(Y) = I(X) + (t_1^{q-1} - t_{s+1}^{q-1})$ , then*

- (a)  $F_Y(t) = F_X(t)(1 + t + \dots + t^{q-2})$ ,
- (b)  $|Y| = (q-1)|X|$ ,
- (c)  $\text{reg}(S[u]/I(Y)) = (q-2) + \text{reg}(S/I(X))$ , where  $u = t_{s+1}$ .

*Proof.* As  $I(X)$  and  $I(Y)$  are lattice ideals [25, Theorem 2.1],  $t_i$  is not a zero divisor of  $S/I(X)$  (resp.  $S[u]/I(Y)$ ) for  $1 \leq i \leq s$  (resp.  $1 \leq i \leq s+1$ ). Hence, there are exact sequences

$$\begin{aligned} 0 \longrightarrow S[u]/I(Y)[-1] &\xrightarrow{u} S[u]/I(Y) \longrightarrow S[u]/(u, I(Y)) \longrightarrow 0, \\ 0 \longrightarrow S/I(X)[- (q-1)] &\xrightarrow{t_1^{q-1}} S/I(X) \longrightarrow S/(t_1^{q-1}, I(X)) \longrightarrow 0. \end{aligned}$$

Therefore, using that  $S[u]/(u, I(Y)) = S[u]/(u, t_1^{q-1}, I(X)) \simeq S/(t_1^{q-1}, I(X))$ , we get

$$F_Y(t) = tF_Y(t) + F(t) \quad \text{and} \quad F_X(t) = t^{q-1}F_X(t) + F(t),$$

where  $F(t)$  is the Hilbert series of  $S/(t_1^{q-1}, I(X))$ . Part (a) follows readily from these two equations. Recall that  $S[u]/I(Y)$  is also a Cohen-Macaulay standard algebra of dimension 1 [11]. Therefore, there are unique polynomials  $g_Y(t)$  and  $g_X(t)$  in  $\mathbb{Z}[t]$  such that

$$F_Y(t) = g_Y(t)/(1-t) \quad \text{and} \quad F_X(t) = g_X(t)/(1-t).$$

Hence, from (a), we get

$$(2.2) \quad g_Y(t) = g_X(t)(1 + t + \dots + t^{q-2}).$$

Making  $t = 1$  in Eq. (2.2), we obtain

$$|Y| = g_Y(1) = (q-1)g_X(1) = (q-1)|X|.$$

This proves (b). Part (c) follows from Eq. (2.2) because  $\text{reg}(S[u]/I(Y))$  is the degree of the polynomial  $g_Y(t)$  and  $\text{reg}(S/I(X))$  is the degree of the polynomial  $g_X(t)$ .  $\square$

**Lemma 2.13.** *Let  $X \subset \mathbb{P}^{s-1}$  and  $X' \subset \mathbb{P}^{s'-1}$  be algebraic toric sets parameterized by  $y^{v_1}, \dots, y^{v_s}$  and  $y^{v_1}, \dots, y^{v_{s'}}$  respectively. If  $s \leq s'$  and  $|X| = |X'|$ , then  $\text{reg } S'/I(X') \leq \text{reg } S/I(X)$ , where  $S' = K[t_1, \dots, t_{s'}]$ .*

*Proof.* Using that  $I(X)$  and  $I(X')$  are vanishing ideals generated by homogeneous polynomials, it is not hard to show that  $S \cap I(X') = I(X)$ . Hence, we have an inclusion of graded modules:

$$S/I(X) \hookrightarrow S'/I(X').$$

Thus,  $H_X(d) \leq H_{X'}(d)$  for  $d \geq 0$ . Recall that  $H_X(d) = |X|$  and  $H_{X'}(d) = |X'|$  for  $d \gg 0$  [20]. Therefore, taking into account that  $|X| = |X'|$ , we obtain:

$$\text{reg } S'/I(X') \leq \text{reg } S/I(X),$$

as required.  $\square$

**Remark 2.14.** As  $S/I(X)$  and  $S'/I(X')$  are Cohen-Macaulay rings of the same dimension and of the same degree (multiplicity), Lemma 2.13 can also be shown using [5, Proposition 3.1].

**Definition 2.15.** The algebraic toric set  $\mathbb{T} = \{[(x_1, \dots, x_s)] \in \mathbb{P}^{s-1} \mid x_i \in K^* \text{ for all } i\}$  is called a *projective torus* in  $\mathbb{P}^{s-1}$ .

**Proposition 2.16.** [15, Theorem 1, Lemma 1] *If  $\mathbb{T}$  is a projective torus in  $\mathbb{P}^{s-1}$ , then*

- (a)  $I(\mathbb{T}) = (t_1^{q-1} - t_s^{q-1}, t_2^{q-1} - t_s^{q-1}, \dots, t_{s-1}^{q-1} - t_s^{q-1})$ .
- (b)  $F_{\mathbb{T}}(t) = (1 - t^{q-1})^{s-1}/(1 - t)^s$ .
- (c)  $\text{reg}(S/I(\mathbb{T})) = (s-1)(q-2)$  and  $\deg(S/I(\mathbb{T})) = (q-1)^{s-1}$ .

**Definition 2.17.** Let  $G$  be a bipartite graph with bipartition  $(V_1, V_2)$ . If every vertex in  $V_1$  is joined to every vertex in  $V_2$ , then  $G$  is called a *complete bipartite graph*. If  $V_1$  and  $V_2$  have  $s_1$  and  $s_2$  vertices respectively, we denote a complete bipartite graph by  $\mathcal{K}_{s_1, s_2}$ . A *spanning subgraph* of a graph  $G$  is a subgraph containing all the vertices of  $G$ .

**Theorem 2.18.** *Let  $G$  be a connected bipartite graph with bipartition  $(V_1, V_2)$  and let  $X$  be the projective algebraic toric set parameterized by the edges of  $G$ . If  $|V_2| \leq |V_1|$ , then*

$$(|V_1| - 1)(q - 2) \leq \text{reg } S/I(X) \leq (|V_1| + |V_2| - 2)(q - 2).$$

*Furthermore, equality on the left occurs if  $G$  is a complete bipartite graph and equality on the right occurs if  $G$  is a tree.*

*Proof.* We set  $|V_i| = s_i$  for  $i = 1, 2$ . First we prove the inequality on the left. Let  $X_1 \subset \mathbb{P}^{s_1-1}$  and  $X_2 \subset \mathbb{P}^{s_2-1}$  be two projective torus and let  $X' \subset \mathbb{P}^{s_1 s_2 - 1}$  be the algebraic toric set parameterized by the edges of the complete bipartite graph  $\mathcal{K}_{s_1, s_2}$  with bipartition  $(V_1, V_2)$ . According to [14] the corresponding Hilbert functions are related by

$$H_{X'}(d) = H_{X_1}(d)H_{X_2}(d) \text{ for } d \geq 0.$$

By Proposition 2.16(c) the regularity index of  $K[t_1, \dots, t_{s_i}]/I(X_i)$  is equal to  $(s_i - 1)(q - 2)$ . Hence, the regularity index of  $K[t_1, \dots, t_{s_1 s_2}]/I(X')$  is equal to  $(s_1 - 1)(q - 2)$ . Therefore, taking into account that  $|X| = |X'| = (q - 1)^{s_1 + s_2 - 2}$  [25], by Lemma 2.13 we obtain:

$$(s_1 - 1)(q - 2) = \text{reg } K[t_1, \dots, t_{s_1 s_2}]/I(X') \leq \text{reg } S/I(X),$$

as required.

Next, we prove the inequality on the right. Let  $H$  be an spanning tree of  $G$ , that is,  $H$  is a subgraph of  $G$  such that  $H$  is a tree that contains every vertex of  $G$ . Consider the projective algebraic toric set  $X_3$  parameterized by the edges of  $H$ . We may assume that  $v_1, \dots, v_{s_1+s_2-1}$  are the characteristic vectors of the edges of  $H$ . As  $H$  is a tree, by [25, Corollary 3.8], one has  $|X_3| = (q-1)^{s_1+s_2-2}$ . Since  $X_3$  is contained in a projective torus  $\mathbb{T}'$  in  $\mathbb{P}^{s_1+s_2-2}$  and since  $|\mathbb{T}'| = (q-1)^{s_1+s_2-2}$ , we get that  $X_3 = \mathbb{T}'$ , that is,  $X_3$  is a projective torus in  $\mathbb{P}^{s_1+s_2-2}$ . Therefore, by Proposition 2.16(c), we obtain

$$(2.3) \quad \text{reg } K[t_1, \dots, t_{s_1+s_2-1}]/I(X_3) = (s_1 + s_2 - 2)(q - 2),$$

Using [25, Corollary 3.8], we get that  $|X|$  and  $|X_3|$  are equal to  $(q-1)^{s_1+s_2-2}$ . Then, by Lemma 2.13 and Eq. (2.3), we get

$$\text{reg } S/I(X) \leq \text{reg } K[t_1, \dots, t_{s_1+s_2-1}]/I(X_3) = (s_1 + s_2 - 2)(q - 2),$$

as required.  $\square$

A connected graph is always a spanning subgraph of a complete graph. An interesting open problem is to compute the regularity of  $S/I(X)$  for a complete graph because—using Lemma 2.13 and [25, Corollary 3.8]—this would give an optimal lower bound for the regularity of any connected non-bipartite graph (see the proof of Theorem 2.18).

For even cycles the regularity of  $S/I(X)$  and the basic parameters of parameterized codes over even cycles were studied in [13].

**Corollary 2.19.** [13, Corollary 3.1] *If  $G$  is an even cycle of length  $2k$ , then*

$$\text{reg } S/I(X) \geq (k-1)(q-2).$$

*Proof.* If  $(V_1, V_2)$  is the bipartition of  $G$ , then  $|V_1| = |V_2| = k$ . Hence, the inequality follows from Theorem 2.18.  $\square$

The reverse inequality is also true but it is much harder to prove.

**Theorem 2.20.** [24] *If  $G$  is an even cycle of length  $2k$ , then*

$$\text{reg}(S/I(X)) \leq (k-1)(q-2).$$

A cycle containing all the vertices of a graph is called a *Hamilton cycle*. A graph containing a Hamilton cycle is called *Hamiltonian*.

**Corollary 2.21.** *If  $G$  is a Hamiltonian bipartite graph with  $2k$  vertices, then*

$$\text{reg}(S/I(X)) = (k-1)(q-2).$$

*Proof.* Let  $(V_1, V_2)$  be the bipartition of  $G$ , let  $H$  be a Hamilton cycle of  $G$ , and let  $\mathcal{K}_{k,k}$  be the complete bipartite graph with bipartition  $(V_1, V_2)$ . Notice that  $H$  is a spanning subgraph of  $G$  and  $G$  is a spanning subgraph of  $\mathcal{K}_{k,k}$ . Therefore, applying Lemma 2.13 together with Theorem 2.18 and Theorem 2.20, the equality follows.  $\square$

The next open problem is known as the Eisenbud-Goto regularity conjecture [10].

**Conjecture 2.22.** *If  $\mathfrak{p} \subset (t_1, \dots, t_s)^2$  is a prime graded ideal of  $S$ , then*

$$\text{reg}(S/\mathfrak{p}) \leq \deg(S/\mathfrak{p}) - \text{codim}(S/\mathfrak{p}).$$

There is a version of this conjecture, for square-free monomial ideals whose Stanley-Reisner complex is connected in codimension 1, that has been shown in [30]. We will show the Eisenbud-Goto regularity conjecture for vanishing ideals over Hamiltonian bipartite graphs.

**Lemma 2.23.** *Let  $k \geq 2$  and  $q \geq 3$  be two integers. Then (i)  $2^{2k-2} \geq (k-1)(k+2)$ , and (ii)  $(q-1)^{2k-2} \geq (k-1)(q+k-1)$ .*

*Proof.* The inequality in (i) follows readily by induction on  $k$ . The inequality in (ii) follows by induction on  $q$  and using (i).  $\square$

**Corollary 2.24.** *If  $G$  is a Hamiltonian bipartite graph, then*

$$\text{reg}(S/I(X)) \leq \deg(S/I(X)) - \text{codim}(S/I(X)).$$

*Proof.* The graph  $G$  has  $s$  edges and  $n$  vertices. Since  $G$  is Hamiltonian and bipartite,  $n = 2k$  for some integer  $k \geq 2$  and  $G$  has a bipartition  $(V_1, V_2)$  with  $|V_i| = k$  for  $i = 1, 2$ . Thus,  $s \leq k^2$ . Hence, by Lemma 2.23, we have:

$$(s-1) + (q-2)(k-1) \leq (k^2-1) + (q-2)(k-1) = (k-1)(q+k-1) \leq (q-1)^{2k-2}.$$

To complete the proof notice that  $\deg S/I(X)$  is  $|X| = (q-1)^{2k-2}$  [25, Corollary 3.8],  $\text{codim } S/I(X)$  is  $s-1$  [11] and  $\text{reg } S/I(X)$  is  $(q-2)(k-1)$  (see Corollary 2.21).  $\square$

**Definition 2.25.** Let  $\mathcal{C}$  be a clutter and let  $y_i$  be a vertex. We say  $y_i$  is a *free vertex* of  $\mathcal{C}$  if  $y_i$  only appears in one of the edges of  $\mathcal{C}$ .

**Definition 2.26.** If  $a \in \mathbb{R}^n$ , its *support* is defined as  $\text{supp}(a) = \{i \mid a_i \neq 0\}$ . The *support* of the monomial  $y^a$  is defined as  $\text{supp}(y^a) = \{y_i \mid a_i \neq 0\}$ .

**Theorem 2.27.** *Let  $\mathcal{C}$  be a clutter and let  $X'$  be the projective algebraic toric set parameterized by  $y^{v_1}, \dots, y^{v_{s-1}}$ . If  $y_n$  is a free vertex of  $\mathcal{C}$  and  $y_n \in \text{supp}(y^{v_s})$ , then*

- (a)  $I(X) = I(X') + (t_1^{q-1} - t_s^{q-1})$ .
- (b)  $\text{reg } S/I(X) = \text{reg } S'/I(X') + (q-2)$ , where  $S' = K[t_1, \dots, t_{s-1}]$ .
- (c)  $\deg S/I(X) = (q-1)\deg S'/I(X')$ .

*Proof.* (a) We set  $I' = I(X') + (t_1^{q-1} - t_s^{q-1})$ . Clearly  $I' \subset I(X)$ . Recall that  $I(X)$  is generated by a finite set of binomials [25]. To show the inclusion  $I(X) \subset I'$  we proceed by contradiction. Pick a homogeneous binomial  $g$  in  $I(X) \setminus I'$  of least possible degree, i.e., any binomial of  $I(X)$  of degree less than  $\deg(g)$  belongs to  $I'$ . We can write

$$g = t_1^{a_1} \dots t_s^{a_s} - t_1^{b_1} \dots t_s^{b_s},$$

with  $\text{supp}(t_1^{a_1} \dots t_s^{a_s}) \cap \text{supp}(t_1^{b_1} \dots t_s^{b_s}) = \emptyset$ . If  $a_s = b_s = 0$ , then  $g \in I(X')$  which is impossible. Thus, we may assume that  $a_s > 0$  and  $b_s = 0$ . Thus,  $b_i > 0$  for some  $1 \leq i \leq s-1$ . For simplicity of notation, we assume that  $i = 1$ . Making  $x_i = 1$  for  $i = 1, \dots, n-1$  in the equality

$$(x^{v_1})^{a_1} \dots (x^{v_s})^{a_s} = (x^{v_1})^{b_1} \dots (x^{v_s})^{b_s}$$

we get  $x_n^{a_s} = 1$  for any  $x_n \in K^*$ . In particular, if  $\beta$  is a generator of the cyclic group  $(K^*, \cdot)$  and  $x_n = \beta$ , we get  $\beta^{a_s} = 1$ . Hence, we can write  $a_s = \mu(q-1)$  for some integer  $\mu$ . As  $b_1 > 0$ , one has the equality

$$\begin{aligned} h &= (t_1^{a_1} \dots t_s^{a_s} - t_1^{b_1} \dots t_s^{b_s}) + (t_1^{q-1} - t_s^{q-1})(t_s^{(\mu-1)(q-1)} t_1^{a_1} \dots t_{s-1}^{a_{s-1}}) \\ &= -t_1^{b_1} \dots t_s^{b_s} + t_1^{q-1} t_s^{(\mu-1)(q-1)} t_1^{a_1} \dots t_{s-1}^{a_{s-1}} = t_1 g_1 \end{aligned}$$

for some binomial  $g_1$ . Notice that  $h \neq 0$ , otherwise  $g \in I'$  which is impossible. Therefore  $g_1$  is in  $I(X) \setminus I'$  and has degree less than  $\deg(g)$ , a contradiction to the choice of  $g$ .

(b) We set  $B = S'/(I(X'), t_1^{q-1})$ . By part (a),  $B = S/(I(X), t_s)$ . There are exact sequences

$$\begin{aligned}
0 \longrightarrow S/I(X)[-1] &\xrightarrow{t_s} S/I(X) \longrightarrow B \longrightarrow 0, \\
0 \longrightarrow S'/I(X')[-(q-1)] &\xrightarrow{t_1^{q-1}} S'/I(X') \longrightarrow B \longrightarrow 0.
\end{aligned}$$

Therefore

$$(1-t)F_X(t) = F(B, t) \quad \text{and} \quad (1-t^{q-1})F_{X'}(t) = F(B, t)$$

where  $F(B, t)$  is the Hilbert series of  $B$ . Thus,  $F_X(t) = (1+t+\dots+t^{q-2})F_{X'}(t)$ . From this equality (b) follows (see the last part of the proof of Proposition 2.12). Part (c) also follows from this equality.  $\square$

A graph with exactly one cycle is called *unicyclic*.

**Corollary 2.28.** *Let  $G$  be a connected bipartite graph. If  $G$  is unicyclic with a cycle of length  $2k$ , then  $\text{reg } S/I(X) = (q-2)(n-k-1)$ .*

*Proof.* We proceed by induction on  $n$ . Notice that  $s = n$ , i.e., the number of edges of  $G$  is equal to the number of vertices of  $G$ . This follows using that  $G$  is connected and unicyclic. If  $G$  is a cycle, then  $s = 2k$  and the result follows from Corollary 2.21. If  $G$  is not a cycle, then  $G$  has a vertex  $y_i$  of degree 1. We may assume that  $y^{v_s}$  is the only monomial that contains  $y_i$ . Hence, by Theorem 2.27, we get

$$\text{reg } S/I(X) = \text{reg } S'/I(X') + (q-2),$$

where  $X'$  is parameterized by  $y^{v_1}, \dots, y^{v_{s-1}}$  and  $S' = K[t_1, \dots, t_{s-1}]$ . To complete the proof notice that by induction hypothesis we have  $\text{reg } S'/I(X') = (n-2-k)(q-2)$ .  $\square$

A graph  $G$  is *chordal* if every cycle of  $G$  of length  $n \geq 4$  has a chord. A *chord* of a cycle is an edge joining two non adjacent vertices of the cycle. If  $v$  is a vertex of a graph  $G$ , then its *neighbor set*, denoted by  $N_G(v)$ , is the set of vertices of  $G$  adjacent to  $v$ . Let  $U$  be a set of vertices of  $G$ . The *induced subgraph*  $G[U]$  is the maximal subgraph of  $G$  with vertex set  $U$ .

**Lemma 2.29.** [31, Theorem 8.3] *Let  $G$  be a chordal graph and let  $\mathcal{K}$  be a complete subgraph of  $G$ . If  $\mathcal{K} \neq G$ , then there is  $v \notin V_{\mathcal{K}}$  such that  $G[N_G(v)]$  is a complete subgraph.*

Let  $G$  be a graph. A *clique* of  $G$  is a set of mutually adjacent vertices. The *clique clutter* of  $G$ , denoted by  $\text{cl}(G)$ , is the clutter on  $V_G$  whose edges are the maximal cliques of  $G$  (maximal with respect to inclusion). Given  $v \in V_G$ , by  $G \setminus \{v\}$ , we mean the graph formed from  $G$  by deleting  $v$ , and all edges incident to  $v$ .

**Corollary 2.30.** *Let  $\mathcal{C} = \text{cl}(G)$  be the clique clutter of a chordal graph  $G$ . If  $\mathcal{C}$  has  $s$  edges, then  $\text{reg } S/I(X) = (q-2)(s-1)$  and  $\deg S/I(X) = (q-1)^{s-1}$ .*

*Proof.* From Lemma 2.29, the clique clutter of  $G$  has a free vertex. We denote this vertex by  $y_n$ . Using that  $G \setminus \{y_n\}$  is a chordal graph, together with Theorem 2.27, the result follows by induction on the number of edges of  $\text{cl}(G)$ .  $\square$

**Corollary 2.31.** *If  $G$  is a connected bipartite graph with a largest cycle of length  $2k$ , then*

$$\text{reg } S/I(X) \leq (q-2)(n-k-1).$$

*Proof.* Let  $C$  be a cycle of  $G$  of length  $2k$ . It is not hard to see, by induction on the number of vertices, that  $G$  has a unicyclic connected subgraph  $H$  with  $V_G = V_H$  and whose only cycle is  $C$ . Let  $X'$  be the set parameterized by the edges of  $H$ . We set  $S' = K[t_1, \dots, t_n]$ , where  $t_1, \dots, t_n$  are the variables that correspond to the monomials defining the edges of  $H$ . By Corollary 2.28,  $\text{reg}(S'/I(X'))$  is equal to  $(q-2)(n-k-1)$ . Notice that  $|X| = |X'| = (q-1)^{n-2}$  because  $H$  and  $G$  are both connected bipartite graphs with  $n$  vertices (see [25, Corollary 3.8]). Thus, by Lemma 2.13,  $\text{reg}(S/I(X)) \leq \text{reg}(S'/I(X'))$ . This proves the required inequality.  $\square$

**Example 2.32.** Let  $K = \mathbb{F}_3$  be the field with 3 elements and let  $X$  be the projective algebraic toric set parameterized by the monomials:

$$x_1x_6, x_1x_2, x_1x_8, x_3x_2, x_3x_4, x_5x_6, x_5x_4, x_5x_8, x_7x_2, x_7x_4.$$

The graph  $G$ , whose edges correspond to these monomials, is connected and bipartite with bipartition  $V_1 = \{x_1, x_3, x_5, x_7\}$ ,  $V_2 = \{x_6, x_2, x_4, x_8\}$ . All vertices of this graph have degree at least two. The largest cycle of  $G$  has length 6. Thus, by a direct application of Corollary 2.31 and Theorem 2.18, we get  $3 \leq \text{reg}(S/I(X)) \leq 4$ . Using *Macaulay2* [17] it is seen that the regularity of  $S/I(X)$  is equal to 4.

### 3. APPLICATIONS TO CODING THEORY

We continue to use the notation and definitions used in Sections 1 and 2. In this section we recall the well known interconnections between the algebraic invariants of vanishing ideals and the basic parameters of affine and projective parameterized linear codes. Then we present upper and lower bounds for the minimum distance of parameterized codes arising from connected bipartite graphs.

Some families of evaluation codes have been studied extensively using commutative algebra methods and especially Hilbert functions, see [7, 8, 12, 16, 25, 28]. In this section we use these methods to study parameterized codes over finite fields.

Let  $S = K[t_1, \dots, t_s] = \bigoplus_{d=0}^{\infty} S_d$  be a polynomial ring over the field  $K$  with the standard grading, let  $Q_1, \dots, Q_r$  be the points of  $X^*$ , and let  $S_{\leq d}$  be the set of polynomials of  $S$  of degree at most  $d$ .

**Definition 3.1.** The *evaluation map*  $\text{ev}_d: S_{\leq d} \rightarrow K^{|X^*|}$ ,  $f \mapsto (f(Q_1), \dots, f(Q_r))$ , defines a  $K$ -linear map. The image of  $\text{ev}_d$ , denoted by  $C_{X^*}(d)$ , is a *linear code* which is called a *parameterized affine code* of degree  $d$  on  $X^*$ , by a *linear code* we mean a linear subspace of  $K^{|X^*|}$ .

Parameterized affine codes are special types of Reed-Muller codes (in the sense of [33, p. 37]) and evaluation codes [8, 12, 16, 21]. If  $s = n = 1$  and  $v_1 = 1$ , then  $X^* = \mathbb{F}_q^*$  and we obtain the classical Reed-Solomon code of degree  $d$  [33, p. 33].

The *dimension* and the *length* of  $C_{X^*}(d)$  are given by  $\dim_K C_{X^*}(d)$  and  $|X^*|$  respectively. The dimension and the length are two of the *basic parameters* of a linear code. A third basic parameter is the *minimum distance* which is given by

$$\delta_{X^*}(d) = \min\{\|v\| : 0 \neq v \in C_{X^*}(d)\},$$

where  $\|v\|$  is the number of non-zero entries of  $v$ . The basic parameters of  $C_{X^*}(d)$  are related by the *Singleton bound* for the minimum distance:

$$\delta_{X^*}(d) \leq |X^*| - \dim_K C_{X^*}(d) + 1.$$

Two of the parameters of  $C_{X^*}(d)$  can be expressed using Hilbert functions of standard graded algebras as is seen below.

**Definition 3.2.** The evaluation map

$$\text{ev}'_d: S[u]_d \rightarrow K^{|Y|}, \quad f \mapsto \left( \frac{f(Q_1, 1)}{f_0(Q_1, 1)}, \dots, \frac{f(Q_r, 1)}{f_0(Q_r, 1)} \right),$$

where  $f_0(t_1, \dots, t_{s+1}) = t_1^d$ , defines a linear map of  $K$ -vector spaces. The image of  $\text{ev}'_d$ , denoted by  $C_Y(d)$ , is called a *parameterized projective code* of degree  $d$  on the set  $Y$ . The minimum distance of  $C_Y(d)$  is denoted by  $\delta_Y(d)$ .

**Definition 3.3.** The *affine Hilbert function* of  $S/I(X^*)$  is given by

$$H_{X^*}(d) := \dim_K S_{\leq d}/I(X^*)_{\leq d}$$

where  $I(X^*)_{\leq d} = S_{\leq d} \cap I(X^*)$ .

This paper is motivated by the fact that the degree and the Hilbert function of  $S[u]/I(Y)$  are related to the basic parameters of parameterized affine linear codes:

**Theorem 3.4.** [22, Theorem 2.4] (a)  $C_{X^*}(d) \simeq C_Y(d)$  as  $K$ -vector spaces.

(b) The parameterized codes  $C_{X^*}(d)$  and  $C_Y(d)$  have the same parameters.

(c) The dimension and the length of  $C_{X^*}(d)$  are  $H_Y(d)$  and  $\deg(S[u]/I(Y))$  respectively.

(d) [22, Remark 2.5]  $H_Y(d) = H_{X^*}(d)$  for  $d \geq 0$  (cf. [18, Remark 5.3.16]).

**Lemma 3.5.** Let  $X \subset \mathbb{P}^{s-1}$  and  $X' \subset \mathbb{P}^{s'-1}$  be algebraic toric sets parameterized by  $y^{v_1}, \dots, y^{v_s}$  and  $y^{v_1}, \dots, y^{v_{s'}}$  respectively. If  $s \leq s'$  and  $|X| = |X'|$ , then  $\delta_{X'}(d) \leq \delta_X(d)$ .

*Proof.* We can choose  $P_1, \dots, P_m$  in  $X^*$  so that  $X = \{[P_1], \dots, [P_m]\}$ . There is a well defined epimorphism

$$\phi: X' \rightarrow X, \quad [(x^{v_1}, \dots, x^{v_{s'}})] \mapsto [(x^{v_1}, \dots, x^{v_s})],$$

induced by the map  $[(\alpha_1, \dots, \alpha_{s'})] \mapsto [(\alpha_1, \dots, \alpha_s)]$ . By hypothesis  $|X'| = |X|$ . Hence, the map  $\phi$  is an isomorphism of multiplicative groups. Thus, we can write  $X' = \{[P'_1], \dots, [P'_m]\}$  so that  $[P'_i]$  maps under  $\phi$  to  $[P_i]$  for all  $i$ . Pick  $F$  in  $S_d = K[t_1, \dots, t_s]_d$  such that  $\delta_X(d) = |\{P_i \mid F(P_i) \neq 0\}|$ . Notice that  $F$  is also a polynomial in  $S[t_{s+1}, \dots, t_{s'}]_d$ . Since  $F(P_i) \neq 0$  if and only if  $F(P'_i) \neq 0$ , we get  $\delta_{X'}(d) \leq \delta_X(d)$ .  $\square$

**Theorem 3.6.** Let  $G$  be a connected bipartite graph with bipartition  $(V_1, V_2)$ . Then

$$\delta_{X_1}(d)\delta_{X_2}(d) \leq \delta_X(d) \leq \delta_{X_3}(d), \quad \text{for } d \geq 1,$$

where  $X_3$  is a projective torus in  $\mathbb{P}^{|V_1|+|V_2|-2}$  and  $X_i$  is a projective torus in  $\mathbb{P}^{|V_i|-1}$  for  $i = 1, 2$ .

*Proof.* We set  $|V_i| = s_i$  for  $i = 1, 2$ . First we prove the inequality on the left. Let  $X' \subset \mathbb{P}^{s_1 s_2 - 1}$  be the projective algebraic toric set parameterized by the edges of the complete bipartite graph  $\mathcal{K}_{s_1, s_2}$  with bipartition  $(V_1, V_2)$ . According to [14] the minimum distances are related by  $\delta_{X'}(d) = \delta_{X_1}(d)\delta_{X_2}(d)$  for  $d \geq 1$ . Recall that  $|X| = |X'| = (q-1)^{s_1+s_2-2}$  [25]. Therefore, by Lemma 3.5, we obtain the inequality on the left.

Let  $H$  be a spanning tree of  $G$ , that is,  $H$  is a subgraph of  $G$  such that  $H$  is a tree that contains every vertex of  $G$ . Consider the algebraic toric set  $X''$  parameterized by the edges of  $H$ . We may assume that  $v_1, \dots, v_{s_1+s_2-1}$  are the characteristic vectors of the edges of  $H$ . Notice that the set  $X''$  is a projective torus in  $\mathbb{P}^{s_1+s_2-2}$ , i.e.,  $X'' = X_3$  (see the proof of Theorem 2.18). Using [25, Corollary 3.8], we get that  $|X|$  and  $|X_3|$  are equal to  $(q-1)^{s_1+s_2-2}$ . Therefore, by Lemma 3.5, we obtain the inequality on the right.  $\square$

Lower bounds for the minimum distance of evaluation codes have been shown when  $X$  is any complete intersection reduced set of points in a projective space [3, 12, 19], and when  $X$  is a reduced Gorenstein set of points [32]. Upper bounds for the minimum distance of certain parameterized codes are given in [25, 27].

There is a nice recent formula for the minimum distance of a parameterized code over a projective torus.

**Theorem 3.7.** [26, Theorem 3.4] *If  $\mathbb{T}$  is a projective torus in  $\mathbb{P}^{n-1}$  and  $d \geq 1$ , then the minimum distance of  $C_{\mathbb{T}}(d)$  is given by*

$$\delta_{\mathbb{T}}(d) = \begin{cases} (q-1)^{n-(k+2)}(q-1-\ell) & \text{if } d \leq (q-2)(n-1)-1, \\ 1 & \text{if } d \geq (q-2)(n-1), \end{cases}$$

where  $k$  and  $\ell$  are the unique integers such that  $k \geq 0$ ,  $1 \leq \ell \leq q-2$  and  $d = k(q-2) + \ell$ .

**Corollary 3.8.** *Let  $G$  be a connected bipartite graph with bipartition  $(V_1, V_2)$ . Then*

$$(q-2)^2(q-1)^{|V_1|+|V_2|-4} \leq \delta_X(1) \leq (q-2)(q-1)^{|V_1|+|V_2|-3}.$$

*Proof.* It follows readily from Theorems 3.6 and 3.7.  $\square$

**Example 3.9.** Let  $G$  be a cycle of length 6. If  $K = \mathbb{F}_5$ , then  $144 \leq \delta_X(1) \leq 192$ . The exact value of  $\delta_X(1)$  is 186.

**Proposition 3.10.** *Let  $\mathcal{C}$  be a clutter and let  $G$  be a graph. (a) If there is  $C \subset V_{\mathcal{C}}$  so that  $|C \cap e| = 1$  for any  $e \in E_{\mathcal{C}}$ , then  $\delta_Y(d) \leq (q-1)\delta_X(d)$  for any  $d \geq 1$ . (b) If  $G$  is a connected bipartite graph, then  $\delta_Y(1) = (q-1)\delta_X(1)$ .*

*Proof.* We may assume that  $C = \{y_1, \dots, y_\ell\}$ . Let  $\beta$  be a generator of the cyclic group  $(\mathbb{F}_q^*, \cdot)$ . We can choose  $P_1, \dots, P_m$  in  $X^*$  so that  $X = \{[P_1], \dots, [P_m]\}$ . If  $P = P_\ell$  for some  $\ell$ , then we can write  $P = (x^{v_1}, \dots, x^{v_s})$ . We set  $\gamma_i = \beta^i$ . From the equality

$$\gamma_i P = ((\gamma_i x_1)^{v_{11}} \dots (\gamma_i x_\ell)^{v_{1\ell}} x_{\ell+1}^{v_{1,\ell+1}} \dots x_n^{v_{1n}}, \dots, (\gamma_i x_1)^{v_{s1}} \dots (\gamma_i x_\ell)^{v_{s\ell}} x_{\ell+1}^{v_{s,\ell+1}} \dots x_n^{v_{sn}})$$

we get that  $\gamma_i P \in X^*$ . By Proposition 2.5, we have that  $|Y| = (q-1)|X|$ . Therefore

$$Y = \{[(\beta P_1, 1)], \dots, [(\beta^{q-1} P_1, 1)], \dots, [(\beta P_m, 1)], \dots, [(\beta^{q-1} P_m, 1)]\}.$$

To show (a) pick  $0 \neq f \in S_d$  such that  $\delta_X(d) = |\{P_i \mid f(P_i) \neq 0\}|$ . Notice that  $f(P_i) = 0$  if and only if  $f(\beta^j P_i) = 0$  for  $1 \leq j \leq q-1$ . Hence,  $f$  does not vanish in exactly  $(q-1)\delta_X(d)$  points of  $Y$ . Consequently  $\delta_Y(d)$  is at most  $(q-1)\delta_X(d)$ .

To show (b) pick a polynomial  $F$  in  $S[u]_1$  such that  $\delta_Y(1) = |\{Q \in Y \mid F(Q) \neq 0\}|$ . If  $F \in S$ , then  $F(P_i) \neq 0$  if and only if  $F(\beta^j P_i, 1) \neq 0$  for some  $1 \leq j \leq q-1$  if and only if  $F(\beta^j P_i, 1) \neq 0$  for all  $1 \leq j \leq q-1$ . Hence,  $\delta_Y(1) = (q-1)r_0$ , where  $r_0 = |\{P_i \mid F(P_i) \neq 0\}|$ . Thus,  $(q-1)\delta_X(1) \leq (q-1)r = \delta_Y(1)$ . Then, using (a), we get  $\delta_Y(1) = (q-1)\delta_X(1)$ . We may now assume that  $F = \lambda_1 t_1 + \dots + \lambda_s t_s + u$ , where  $\lambda_i \in K$  for all  $i$ . It is not hard to verify that if  $F(\beta^\ell P_i, 1) = 0$  for some  $1 \leq \ell \leq q-1$ , then  $F(\beta^j P_i, 1) \neq 0$  for all  $1 \leq j \leq q-1$ ,  $j \neq \ell$ . Hence, the number of zeros in  $Y$  of  $F$  is at most  $|X| = (q-1)^{n-2}$ . Consequently one has

$$(q-1)^{n-2}(q-2) = |Y| - (q-1)^{n-2} \leq \delta_Y(1) \leq (q-1)\delta_X(1) \leq (q-1)^{n-2}(q-2).$$

The first equality is shown in Corollary 2.9 and the last inequality follows from Corollary 3.8. Therefore, we have equality everywhere. In particular  $\delta_Y(1) = (q-1)\delta_X(1)$ .  $\square$

4. COMPLETE INTERSECTION  $I(Y)$  FROM CLUTTERS

We continue to use the notation and definitions used in Sections 1 and 2. In this section we characterize when  $I(Y)$  is a complete intersection in algebraic and geometric terms. For graphs, we describe in graph theoretical terms and in terms of the number of elements of the base field when  $I(Y)$  is a complete intersection.

**Lemma 4.1.** *Let  $\mathcal{C}$  be a clutter. If  $f \neq 0$  is a homogeneous polynomial of  $I(Y)$  of the form  $t_i^b - t^c$  with  $b \in \mathbb{N}$ ,  $c \in \mathbb{N}^s$  and  $i \notin \text{supp}(c)$ , then  $\deg(f) \geq q - 1$ . Moreover if  $b = q - 1$ , then  $f = t_i^{q-1} - t_j^{q-1}$  for some  $j \neq i$ .*

*Proof.* It follows adapting the proof of [27, Lemma 3.4].  $\square$

**Definition 4.2.** The ideal  $I(Y)$  is called a *complete intersection* if it can be generated by  $s$  homogeneous polynomials of  $S[u]$ .

**Lemma 4.3.** *Let  $\mathcal{C}$  be a clutter. If  $I(Y)$  is a complete intersection, then*

$$I(Y) = (t_1^{q-1} - t_{s+1}^{q-1}, \dots, t_s^{q-1} - t_{s+1}^{q-1}).$$

*Proof.* Taking into account Lemma 4.1, we can use the same proof of [26, Theorem 4.4].  $\square$

**Definition 4.4.** The *projective closure* of  $X^*$ , denoted by  $\overline{X^*}$ , is given by  $\overline{X^*} := \overline{Y}$ , where  $\overline{Y}$  is the closure of  $Y$  in the Zariski topology of  $\mathbb{P}^s$ .

The next theorem complements a result of [26] showing that  $I(X)$  is a complete intersection if and only if  $X$  is a projective torus.

**Theorem 4.5.** *Let  $\mathcal{C}$  be a clutter with  $s$  edges and let  $T = \{(x_1, \dots, x_s) \mid x_i \in K^* \text{ for all } i\}$  be an affine torus in  $\mathbb{A}^s$ . The following are equivalent:*

- (a<sub>1</sub>)  $I(Y)$  is a complete intersection.
- (a<sub>2</sub>)  $I(Y) = (t_1^{q-1} - t_{s+1}^{q-1}, \dots, t_s^{q-1} - t_{s+1}^{q-1})$ .
- (a<sub>3</sub>)  $X^* = T$ .
- (a<sub>4</sub>)  $I(X^*) = (t_1^{q-1} - 1, \dots, t_s^{q-1} - 1)$ .

*Proof.* (a<sub>1</sub>) $\Rightarrow$ (a<sub>2</sub>): It follows at once from Lemma 4.3. (a<sub>2</sub>) $\Rightarrow$ (a<sub>3</sub>): By Proposition 2.16 one has  $I(Y) = I(\mathbb{T}') = (\{t_i^{q-1} - t_{s+1}^{q-1}\}_{i=1}^s)$ , where  $\mathbb{T}'$  is a projective torus in  $\mathbb{P}^s$ . As  $Y$  and  $\mathbb{T}'$  are both projective varieties, we get that  $Y = \mathbb{T}'$  (see [25, Lemma 4.2]). We need only show the inclusion  $T \subset X^*$ . Take  $a$  in  $T$ . Then,  $[(a, 1)] \in \mathbb{T}' = Y$ . Thus, we get  $a \in X^*$ . (a<sub>3</sub>) $\Rightarrow$ (a<sub>4</sub>): We need only show the inclusion “ $\subset$ ”. Take  $f \in I(X^*)$ . By the division algorithm [1, Theorem 1.5.9, p. 30] we can write

$$f = h_1(t_1^{q-1} - 1) + \dots + h_s(t_s^{q-1} - 1) + g,$$

for some  $h_1, \dots, h_s, g$  in  $S$ , where the monomials that occur in  $g$  are not divisible by any of the monomials  $t_1^{q-1}, \dots, t_s^{q-1}$ , i.e.,  $\deg_{t_i}(g) < q - 1$  for  $i = 1, \dots, s$ . Hence, since  $g$  vanishes on all  $T$ , using the Combinatorial Nullstellensatz [2, Theorem 1.2] it follows readily that  $g = 0$ , that is,  $f \in (\{t_i^{q-1} - 1\}_{i=1}^s)$ . (a<sub>4</sub>) $\Rightarrow$ (a<sub>1</sub>): Let  $\succ$  be the *elimination order* on the monomials of  $S[u]$ , where  $u = t_{s+1}$ . Recall that this order is defined as  $t^b \succ t^a$  if the degree of  $t^b$  is greater than that of  $t^a$ , or both degrees are equal, and the last nonzero component of  $b - a$  is negative. As  $K$  is a finite field,  $Y$  is the projective closure of  $X^*$ , i.e.,  $\overline{X^*} = \overline{Y} = Y$ . Since  $t_1^{q-1} - 1, \dots, t_s^{q-1} - 1$  form a Gröbner basis with respect to  $\succ$ , using [34, Proposition 2.4.30], we get the equality  $I(Y) = (\{t_i^{q-1} - t_{s+1}^{q-1}\}_{i=1}^s)$ . Thus  $I(Y)$  is a complete intersection.  $\square$

**Corollary 4.6.** *Let  $\mathcal{C}$  be a clutter. If  $I(Y)$  is a complete intersection, then  $I(X)$  is a complete intersection.*

*Proof.* By Theorem 4.5,  $X^*$  is an affine torus. Then,  $X$  is a projective torus. Consequently  $I(X)$  is a complete intersection by Proposition 2.16.  $\square$

**Corollary 4.7.** *Let  $G$  be a graph. If  $\gcd(q-1, 2) = 1$  or if  $G$  is bipartite, then  $I(Y)$  is a complete intersection if and only if  $I(X)$  is a complete intersection.*

*Proof.* It follows at once from Corollary 2.11 and Corollary 4.6.  $\square$

The converse of Corollary 4.6 is not true as the next example shows.

**Example 4.8.** Let  $X$  be the projective algebraic toric set parameterized  $y_1y_2, y_2y_3, y_1y_3$  and let  $K = \mathbb{F}_5$ . Then,  $I(X) = (t_1^4 - t_3^4, t_2^4 - t_3^4)$  is a complete intersection but

$$I(Y) = (t_3^4 - t_4^4, t_2^2t_3^2 - t_1^2t_4^2, t_1^2t_3^2 - t_2^2t_4^2, t_2^4 - t_4^4, t_1^2t_2^2 - t_3^2t_4^2, t_1^4 - t_4^4)$$

is not a complete intersection. The generators of  $I(Y)$  were computed using the computer algebra system *Macaulay2* [17] and the methods of [22, 25]. If  $K = \mathbb{F}_4$ , then  $I(X)$  and  $I(Y)$  are both complete intersections in concordance with Corollary 4.7.

**Proposition 4.9.** *If  $G$  is a connected graph, then  $I(X)$  is a complete intersection if and only if  $G$  is a tree or  $G$  is a unicyclic graph with a unique odd cycle.*

*Proof.*  $\Rightarrow$  As  $I(X)$  is a complete intersection,  $X \subset \mathbb{P}^{s-1}$  is a projective torus [26, Corollary 4.5]. Thus,  $|X| = (q-1)^{s-1}$ . If  $G$  is bipartite, then  $|X| = (q-1)^{n-2}$  [25, Corollary 3.8]. Hence,  $s = n-1$  and  $G$  is a tree because  $G$  is connected. If  $G$  is not bipartite, then  $|X| = (q-1)^{n-1}$  [25, Corollary 3.8]. Thus,  $s = n$  and  $G$  is a unicyclic graph.

$\Leftarrow$  Let  $\mathbb{T}$  be a projective torus in  $\mathbb{P}^{s-1}$ . If  $G$  is a tree, then  $s = n-1$  and  $|X| = (q-1)^{n-2}$  [25, Corollary 3.8]. Since  $X \subset \mathbb{T}$  and  $|\mathbb{T}| = (q-1)^{s-1}$ , we get that  $|X| = |\mathbb{T}|$ . Thus,  $X = \mathbb{T}$ . Consequently,  $I(X)$  is a complete intersection by Proposition 2.16. If  $G$  is a unicyclic graph with a unique odd cycle, then  $s = n$  and  $|X| = (q-1)^{n-1}$  [25, Corollary 3.8]. Since  $X \subset \mathbb{T}$  and  $|\mathbb{T}| = (q-1)^{s-1}$ , we get that  $|X| = |\mathbb{T}|$ . Thus,  $X = \mathbb{T}$ . Hence,  $I(X)$  is a complete intersection by Proposition 2.16.  $\square$

From this result it follows that for connected graphs, with  $q \geq 3$ , the complete intersection property of  $I(X)$  is independent of the finite field  $K$ . The complete intersection property of  $I(Y)$  depends on the finite field  $K$  as seen in Example 4.8. The following result describes when  $I(Y)$  is a complete intersection for connected graphs.

**Theorem 4.10.** *Let  $G$  be a connected graph. Then  $I(Y)$  is a complete intersection if and only if  $G$  is a tree or  $G$  is a unicyclic graph with a unique odd cycle and  $q$  is even.*

*Proof.*  $\Rightarrow$  By Corollary 4.6,  $I(X)$  is a complete intersection. Then, by Proposition 4.9,  $G$  is a tree or  $G$  is a unicyclic graph with a unique odd cycle. If  $G$  is a tree, there is nothing to prove. Assume that  $G$  is not a tree. Then,  $s = n$ . Notice that in general  $|X^*| = |Y|$ . If  $q$  is odd, then by Corollary 2.9 and Theorem 4.5, we get:

$$|Y| = (q-1)^n/2 \quad \text{and} \quad |Y| = |X^*| = (q-1)^s = (q-1)^n,$$

a contradiction. Thus,  $q$  is even, as required.

$\Leftarrow$  It follows readily from Proposition 4.9 and Corollary 4.7.  $\square$

**Corollary 4.11.** *Let  $G$  be a connected bipartite graph. Then  $I(Y)$  is a complete intersection if and only if  $G$  is a tree.*

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